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FRACTAL GEOMETRY IN STRUCTURES. NUMERICAL METHODS FOR CONVEX ENERGY PROBLEMS

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Abstract—In the present paper certain numerical aspects of the theory of fractals in structures are presented. The fractal geometry is approximated through the I.F.S. (iterated function system) approach or through the F.I. (fractal interpolation). These approximations of the fractal are combined with the methods used in structural analysis in order to calculate stress and strain fields in fractal structures. All types of structures with convex strain energy are studied.

1. INTRODUCTION

The geometry of fractals arises when one wants to describe accurately the geometry of nature. This geometry is central to the various fields of science such as in chemistry, physics, biology and economy. In structural analysis and in applied mechanics, we very often have fractal geometries (Mandelbrot, 1972; Falconer, 1985; Barnsley, 1988; Feder, 1988; Peitgen and Saupe, 1988; Prusinkiewicz and Hanan, 1989; Scholz and Mandelbrot, 1989; Takayasu, 1990; L. Mehauté, 1990; Fleischmann, 1990; Muller and Reinhardt, 1990; Bressloff and Stark, 1991; Bunde and Havlin, 1991; Crilly *et al.*, 1991). We could mention here the crack interfaces in natural bodies (Takayasu, 1990; Scholz and Mandelbrot, 1989), the free surfaces and the interfaces of fractured bones, metals and rocks, the geometry of metallic surfaces after sandblasting (L. Mehauté, 1990) or after a meteoritic rain, the crushed interfaces in composite and granular materials (Bunde and Havlin, 1991), the geometry of diffusion fronts (Feder, 1988), for example, between a metallic and a ceramic material, of percolation patterns, of phase transition regions and of sponged materials (Crilly *et al.*, 1991), and finally the geometry of fluvial systems, arterial systems, nervous cells (Muller and Reinhardt, 1990) and plants (Prusinkiewicz and Hanan, 1989).

In the paper we present here, it is assumed that all the fractals considered are obtained by an appropriately defined iterated function system (I.F.S.) or that they are the result of fractal interpolation (F.I.) of a given set of data. Thus, in both cases the fractal results as the fixed point of a given set of functions. The fixed point approximation, which is equivalent to the approximation of the fractal by classical curves or surfaces (i.e. curves and surfaces of integer Hausdorff dimensions), is combined with the numerical techniques of structural analysis. This combination gives as a result the proposed method for the calculation fractal structures. The assumption we make here is that both a structure Ω and its boundary Γ may have a fractal geometry. The theory of the fractal geometry in structures and solids have been developed by Panagiotopoulos (1992a, b) where we refer the reader for the more theoretical aspects.

In the present paper we deal with fractal bilateral and unilateral structures. As bilateral (resp. unilateral) structures are called those structures for which the principle of virtual or of complementary virtual work holds in equality form (resp. inequality form). This equality form results in the case where classical boundary conditions and constraints hold, whereas the inequality form results from inequality constraints, for example from monotone possibly multivalued (i.e. with complete vertical branches) material laws and boundary conditions, which result from convex generally nonsmooth and nonfinite strain energy functions everywhere. The bilateral problems give rise to the variational equalities which, after discretization, lead to the linear or nonlinear equations of the classical structural analysis; the

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unilateral problems considered here lead to variational inequalities or, after discretization, to convex programming problems with or without inequality constraints. For a complete study of the theory of unilateral problems leading to variational inequalities we refer to Panagiotopoulos (1985). The present paper does not deal with structural analysis problems with nonconvex energy functions leading to hemivariational inequalities (i.e. with non-monotone, possibly multivalued, stress–strain laws and/or boundary conditions) (Moreau and Panagiotopoulos, 1989; Moreau *et al.*, 1988). Such types of problems (e.g. in composites and in reinforced materials) are connected with fractal stress–strain laws and will be the subject of a future paper. The method presented here consists of the approximation of the fractal boundary, domain or interface through classical boundaries, domains or interfaces and of the solution of each one of the resulting classical problems by a classical structural analysis method. This procedure is repeated several times and at the limit the solution of the fractal problem is obtained. Some simple mathematical proofs concerning this convergence are given and then some numerical examples are explained. The first example concerns a fractal tree-like completely irregular network which represents a Golgi cell of the human brain. In the second example, a dynamic problem of a spongy body idealized through the Sierpinski triangle is studied. In the third example, a multifractured plane elastic structure (e.g. a fractured bone) is studied on the assumptions that unilateral contact and Coulomb fraction conditions appear at the interface. Finally, the fourth example deals with the large displacement and large strain problem of a punch, with fractal geometry (e.g. a fractured metal piece or a metal piece after sandblasting) in unilateral geometric nonlinear contact with a rigid support (i.e. contact with the possibility of debonding).

As has been remarked in Panagiotopoulos (1992b), the F.I. method can also be applied for the approximation of complicated curves or surfaces having integer Hausdorff dimensions. Thus, the methods presented here have a broader validity. Still there are many open questions on this subject even for facts which, in the case of integer dimension, would be considered as obvious. Certain elementary relations from the theory of norms are used for the proof of the propositions. The reader who is not familiar with general norms should understand everything in the sense of classical Euclidean R^n-norm. In the present paper, following Panagiotopoulos (1992b) we only use the deterministic I.F.S. which seems to be sufficient for our purposes. The fractal surfaces and interfaces in the numerical examples can also be generated by using random I.F.S. (see e.g. Barnsley, 1988; Bressloff and Stark, 1991). This way of fractal generation leads to a slight modification of the computer codes used; for the needs of solids mechanics, i.e. calculations with fractal geometries etc., the consideration of deterministic fractals seems to be sufficient, at least for the present.

2. THE CONVERGENCE OF THE DISCRETIZATION PROCEDURE FOR FRACTAL STRUCTURES: BILATERAL PROBLEMS

Let Ω be a body and let Γ be its boundary. It is assumed here that both Ω and Γ are fractals and the corresponding structure is denoted by $\{\Omega, \Gamma\}$. According to Panagiotopoulos (1992b), there exist sequences $\{\Gamma_j\}$ and $\{\Omega_j\}$ of classical geometric elements such that $\Gamma_j \to \Gamma$ and $\Omega_j \to \Omega$ as $j \to \infty$ in the sense of Hausdorff metric. For the structure subjected to an incremental procedure (e.g. in the case of large displacements) let us denote by X_j the stress increment $\{\sigma_j\}$, and/or the displacement field increment $\{u_j\}$ of the structure $\{\Omega_j, \Gamma_j\}$. Then $X = \lim X_j$ as $j \to \infty$ is defined to be the solution of the initial problem $\{\Omega, \Gamma\}$ in the increment under consideration. Let us assume that within each increment, X_j is for every j the solution of the equation

$$\dot{W}(G_j)X_j = \dot{p}(G_j), \quad G_j = \{\Omega_j, \Gamma_j\}, \tag{1}$$

where $\dot{p}(G_j)$ is the loading increment of the structure and $\dot{W}(G_j)$ denotes the linearized or tangential operator of the nonlinear problem. As in Panagiotopoulos (1992b), we may assume that $\dot{W}(G_j)$ acts on a Hilbert space V and takes values in the same Hilbert space V for all j. We may assume, for instance, that V is a space defined on $\tilde{\Omega}$, where $\tilde{\Omega}$ contains all the Ω_js and their limit Ω, and the same happens for the boundary $\tilde{\Gamma}$ of $\tilde{\Omega}$ with respect to the Γ_js and the Γ. Then the operators $\dot{W}(G_j)$ and the unknown quantities X_j are appropriately extended on the sets $\tilde{\Omega}-\Omega$. Note that $\tilde{\Omega}$ may coincide with Ω, but in this case we have to define the space V on a fractal set following the heavy mathematical methods of Panagiotopoulos (1985). Another possibility would be to define each operator $\dot{W}(G_j)$ simply on the Hilbert space V_j defined on Ω_j and then to assume that the quantities (norms, linear functionals, operators) defined on V_j tend in some sense to the respective quantities defined on Ω. This method is applied both in the case of a fractal boundary and in the case of a fractal domain, e.g. a Sierpinski gasket. Here all the mathematical technicalities are avoided and it is simply assumed that the whole procedure takes place in the same function space for all subproblems considered. This is well justified from the standpoint of mechanics by modifying the domains Ω_j appropriately through parts of zero modulus of elasticity.

The next proposition concerns the approximation of the fractal structure problem by a finite dimensional discretization scheme, i.e. the finite element (F.E.) method. Let us denote by the upper index h, which is destined to tend to infinity, the corresponding discretized problem which results from the initial one by considering, for example, an appropriate F.E. scheme. We denote by V^h the finite dimensional subspace of V in which the solution of the discretized problem G_j^h (with respect to G^h) corresponding to $G_j = \{\Omega_j, \Gamma_j\}$ (with respect to $G = \{\Omega, \Gamma\}$) has to be sought. Additional general propositions will be proved. They are appropriately formulated for the theoretically oriented engineer, thus avoiding unnecessary mathematical complications. However, if one wants to make the proofs perfect from the mathematical point of view, he has to adapt them to the functional framework introduced by Jonson and Wallin (1984). We denote here by $\|.\|$ and $(.,.)$ the norm and the inner product, respectively, on V.

Proposition 2.1

Suppose that $\dot{W}(G_j)\colon V \to V$ is a linear, bounded and symmetric operator on the Hilbert space V. Moreover, let $\dot{p}(G_j) \in V$ for every j. We assume that:

(i) For every j

$$(\dot{W}(G_j)X^*, X^*) \geqslant c_j \|X^*\|^2 \quad \forall X^* \in V, \tag{2}$$

where c_j is a positive constant depending on $\{G_j\}$ such that for every j:

$$\alpha < c_j, \quad \alpha = \text{constant} > 0.$$

(ii) $p(G_j) \to p(G)$ strongly in V as $j \to \infty$. (3)

(iii) $\dot{W}(G_j)X^* \to \dot{W}(G)X^*$ strongly in V as $j \to \infty$. (4)

(iv) Let $\dot{W}^h(G_j)X_j^h = p^h(G_j)$ in the V^h (5)

be the discretized problem corresponding to eqn (1) and suppose that (i) and (iii) hold for $\dot{W}^h$ in V^h, (ii) holds for $\dot{p}^h$, and that

$$\|X_j^h - X_j\| \to 0 \quad \text{as } h \to \infty. \tag{6}$$

Then

$$X_j^h \to X \quad \text{as } h \to \infty \text{ and } j \to \infty \tag{7}$$

and

$$X^h \to X \quad \text{as } h \to \infty, \tag{8}$$

where X is the solution of the problem $\dot{W}(G)X = \dot{p}(G)$ and X^h the solution of its corresponding finite element version.

Proof. We refer to the proof of proposition 1 (Section 5) of Panagiotopoulos (1992b). As in that proof, we here obtain, from eqn (2) ÷ eqn (4), that

$$X_j \to X \quad \text{strongly in } V \tag{9}$$

and

$$X_j^h \to X^h \quad \text{strongly in } V^h. \tag{10}$$

Indeed we have that

$$\alpha\|X_j\|^2 \leqslant c_j\|X_j\|^2 \leqslant (\dot{W}(G_j)X_j, X_j) = (p(G_j), X_j) \leqslant c\|X_j\|, \tag{11}$$

where the last inequality results from the boundedness of $p(G_j)$. Accordingly $\|X_j\| < c$ and thus $X_j \to X$ weakly in V. We shall prove that X is the unique solution of the problem

$$\dot{W}(G)X = \dot{p}(G). \tag{12}$$

Because of the symmetry of $\dot{W}(G_j)$ we can write for every $X^* \in V$

$$(\dot{p}(G_j), X^*) = (\dot{W}(G_j)X_j, X^*) = (\dot{W}(G_j)X^*, X_j) \to (\dot{W}(G)X^*, X) = (\dot{W}(G)X, X^*). \tag{13}$$

Then eqn (13) with eqn (3) imply that X is a solution of eqn (12). This solution is unique because $\dot{W}(G)$ satisfies an inequality analogous to eqn (2) on V, as can be easily obtained from eqns (2) and (4). To show it we assume the contrary, i.e. that X_1 and X_2 are solutions of eqn (12) and using the aforementioned inequality, we obtain that $\|X_1 - X_2\|^2 \leqslant 0$, i.e. that $X_1 = X_2$. From further argument, as in eqn (5.8) of Panagiotopoulos (1992b), we obtain the strong convergence of eqn (9). This proof repeated for $\dot{W}^h$, X_j^h, $\dot{p}^h$ and V_j^h implies the relationship in eqn (10). Now, one may write that

$$\|X - X_j^h\| = \|X - X_j + X_j - X_j^h\| \leqslant \|X - X_j\| + \|X_j - X_j^h\|. \tag{14}$$

The right hand side (r.h.s.) tends to zero due to eqns (6) and (9) and thus eqn (7) is shown to be true. Furthermore, we have that

$$\|X - X^h\| = \|X - X_j^h + X_j^h - X^h\| \leqslant \|X - X_j^h\| + \|X_j^h - X^h\|. \tag{15}$$

The r.h.s. tends to zero as $j \to \infty$ due to eqns (10) and (7). Thus eqn (8) is proved.

The previous proposition expressed in the language of mechanics takes the following form.

Proposition 2.2

Within each increment the solution of the discretized problem G_j^h tends to the solution of the fractal problem G, as $h \to \infty$ and $j \to \infty$. Moreover, the solution of the discretized problem G^h tends to the solution of the fractal problem G, as $h \to \infty$.

Let us consider now a large displacement, and/or a large strain problem, or a dynamic problem. It is well-known that the solution of these problems denoted here by $\hat{X}$ is obtained by formulating the corresponding incremental problem. Then the solution of the incremental problem X within each increment yields at the limit (e.g. the subdivision of the time interval tends to zero) and in a norm appropriate for the problem under consideration, the solution $\hat{X}$ of the initial problem. Let us write it symbolically as

$$|||\hat{X} - f(X)||| \to 0 \tag{16}$$

where f is an operator (e.g. summation operator) such that $f(X_j) \to f(X)$ as $j \to \infty$ ("continuity" property). Then the following proposition holds.

Proposition 2.3

If the incremental procedure converges for the structure G_j and for the fractal structure G, then the solution $\hat{X}$ of the fractal structure is obtained as the limit for $j \to \infty$ of the solution $\hat{X}_j$ of the nonfractal structure G_j.

Proof. According to the assumptions we shall have, by using the appropriate norm, that $|||\hat{X}-f(X)||| \to 0$ and $|||\hat{X}_j-f(X)||| \to 0$. Then, due to the "continuity" of operator f, we have that $|||f(X_j)-f(X)||| \to 0$ and, thus,

$$|||\hat{X}-\hat{X}_j||| = |||\hat{X}-f(X)+f(X)-f(X_j)+f(X_j)-\hat{X}_j||| \leq |||\hat{X}-f(X)|||+|||\hat{X}_j -f(X_j)|||+|||f(X)-f(X_j)||| \to 0 \quad (17)$$

which implies the result.

The assumption of this proposition is a natural self-evident assumption. Indeed, if a dynamic problem for a fractal structure has to be solved, the fractality of the geometry of the body does not affect the validity of the time discretization method applied to the structure under consideration.

On closing this section, we should notice that the propositions given lead to some general conclusions useful for our calculation needs. We have kept the necessary mathematical tools to a minimum; a more deep mathematical study could give error estimates and more precise and rigorous conclusions from the standpoint of mathematical analysis.

3. UNILATERAL PROBLEMS WITH CONVEX ENERGY

In the previous section we have considered general incremental bilateral problems. Let us consider here a general unilateral problem which covers the case of holonomic plasticity, locking material laws etc., in structures subjected to general monotone, possibly multivalued boundary and/or interface conditions (Panagiotopoulos, 1985) (e.g. plastic hinges in plates, unilateral contact and/or friction in solids). Let $\dot{w}$ be a convex strain energy function such that $\dot{w}(\varepsilon)$ takes values in $(-\infty, +\infty]$, $w(\varepsilon) \not\equiv \infty$, where $\varepsilon = \{\varepsilon_{ik}\}$ $i, k = 1, 2, 3$ is the strain tensor. Note that it is sufficient that $\dot{W}$ be only lower semicontinuous, and not continuous, i.e. it is a superpotential in the sense of Moreau (Panagiotopoulos, 1985; Moreau and Panagiotopoulos, 1989). We introduce the notation

$$\dot{W}(u) = \begin{cases} \int_\Omega \dot{w}(\varepsilon(u))\, d\Omega & \text{if } \dot{w}(\varepsilon(u)) \text{ is integrable,} \\ \infty & \text{otherwise} \end{cases} \quad (18)$$

where u is the displacement vector and we assume that for every $\varepsilon = \{\varepsilon_{ik}\}$ there exists a positive constant

$$\dot{w}(\varepsilon) \geq c\varepsilon_{ik}\varepsilon_{ik}, \quad c = \text{constant} > 0. \quad (19)$$

We again consider an incremental process and for this reason a dot is put on the quantities. Within each increment we have to solve a general unilateral problem of the following type, with respect to the displacements (i.e. in the sequel X and X^* represent displacement increments). Find $X_j \in V$ such as to satisfy the variational inequality

$$\dot{W}(G_j, X^*) - \dot{W}(G_j, X_j) + \Phi(G_j, X^*) - \Phi(G_j, X_j) \geq (\dot{p}(G_j), X^* - X_j) \quad \forall X^* \in V. \quad (20)$$

Here $\Phi(G_j, .)$ is a convex, lower semicontinuous superpotential from V into $(-\infty, +\infty]$ and $\Phi \not\equiv \infty$. $\dot{W}$ has the same properties. The variational inequality (eqn (20)) is a very general variational inequality and covers all the cases of monotone possibly multivalued material laws (e.g. holonomic plasticity with or without hardening, Hencky ideal plasticity without unloading, nonlinear elasticity etc.) combined with all monotone possibly multi-valued boundary laws (e.g. friction laws, unilateral contact laws, plastic boundary zone laws etc.). For the rigorous mathematical study of this variational inequality we refer to Panagiotopoulos (1985), Moreau and Panagiotopoulos (1989) and Moreau *et al.* (1988).

The following proposition holds on the assumption that any rigid body displacement is prevented (e.g. $v = 0$ on an appropriate boundary part of the body etc.) and that there exists a constant $\bar{c}$ independent of G_j such that

$$\bar{c}\|X\|^2 \leqslant \int_{\Omega_j} \varepsilon_{ik}(X)\varepsilon_{ik}(X)\,\mathrm{d}\Omega \quad \forall X \in V. \tag{21}$$

Note that this property always holds if V is a subspace of the Sobolev space $[H^1(\Omega)]^3$ which is the standard space for the rigorous study of the linear elasticity and of the variational inequality [eqn (20)].

Proposition 3.1

Suppose that $\dot{w}$ satisfies eqn (19), where c depends on G_j for every j. Let there be a constant α such that

$$c > \alpha \quad \text{for every } j \tag{22}$$

and let $\dot{p}(G)$ have the property in eqn (3). Moreover let us assume that

$$\Phi(G_j, X) \geqslant -c\|X\|_j - c \quad \forall X \in V_j, \tag{23}$$

where $c \geqslant 0$ independently of G_j for every j; that

$$\liminf \Phi(G_j, X_j) \geqslant \Phi(G, X) \quad \text{for } X_j \to X \text{ weakly in } V; \tag{24}$$

$$\liminf \dot{W}(G_j, X_j) \geqslant \dot{W}(G, X) \quad \text{for } X_j \to X \text{ weakly in } V; \tag{25}$$

$$\dot{W}(G_j, X) \to \dot{W}(G, X) \quad \forall X \in V \quad \text{and} \quad j \to \infty \tag{26}$$

and

$$\Phi(G_j, X) \to \Phi(G, X) \quad \forall X \in V \quad \text{and} \quad j \to \infty. \tag{27}$$

Then $X_j \to X$ weakly in V as $j \to \infty$, where X is a solution of the variational inequality

$$\dot{W}(G, X^*) - \dot{W}(G, X) + \Phi(G, X^*) - \Phi(G, X) \geqslant (\dot{p}(G), X^* - X) \quad \forall X^* \in V. \tag{28}$$

Proposition 3.2

The solution is unique if $\dot{W}$ and/or Φ are strictly convex.

Proof. Suppose now that Φ and/or W are strictly convex. Then the whole potential energy,

$$\Pi(G(X)) = \dot{W}(G, X) + \Phi(G, X) - (\dot{p}(G, X)), \tag{29}$$

becomes minimum over V at the position of equilibrium X, as it results directly from the variational inequality [eqn (28)] which is written as

$$\Pi(G, X) \leqslant \Pi(G, X^*) \quad \forall X^* \in V. \tag{30}$$

Now $\Pi(G, X)$ is strictly convex as a function of X. In order to show the uniqueness of the solution let us assume that X_1 and X_2 are both distinct solutions. From the strict convexity of Π we have that

$$\Pi\left(G, \frac{X_1+X_2}{2}\right) < \tfrac{1}{2}\Pi(G, X_1)+\tfrac{1}{2}\Pi(G, X_2) = \tfrac{1}{2}\left(\min_v \Pi(X^*)+\min_v \Pi(X^*)\right) = \min_v \Pi(X^*) \tag{31}$$

which is a contradiction because $\Pi(G, (X_1+X_2)/2)$ gives a smaller value than the minimum of Π. Thus $X_1 = X_2$, i.e. the uniqueness of the solution results.

Proposition 3.3

If the structure is linear elastic with $C = \{C_{iklm}\}$, the Hooke's tensor is such that

$$\int_{\Omega_j} C_{iklm}\varepsilon_{ik}(X_j)\varepsilon_{lm}(X_j)\,d\Omega \geqslant c\|X_j\|^2 \quad \forall X_j \in V, \tag{32}$$

where c is a constant independent of G_j, then $X_j > X$ strongly in V.

Note that eqn (32) holds if we are in the common functional framework of the Sobolev spaces for an appropriate class of domains Ω_j. Obviously the strong convergence is also guaranteed if $\varepsilon \to \dot{w}(\varepsilon)$ contains a quadratic part and another part expressed through a superpotential Φ_1. Now we can prove some analogous results of proposition 2.1. We consider the discretized form of the variational inequality [eqn (20)] [with respect to eqn (28)], and we denote its solution by X_j^h (with respect to X^h).

Proposition 3.4

Suppose that all the assumptions of proposition 3.1 hold for both the continuous variational inequality problem G_j and the discretized variational inequality problem G_j^h. Moreover, let us assume that the solution X_j^h of the problem G_j^h tends strongly to the solution X_j of the problem G_j as $h \to \infty$, i.e.

$$\|X_j^h - X_j\| \to 0 \quad \text{as } h \to \infty. \tag{33}$$

Then

$$X_j^h \to X \quad \text{weakly as } h \to \infty \quad \text{and} \quad j \to \infty \tag{34}$$

and

$$X^h \to X \quad \text{weakly as } h \to \infty, \tag{35}$$

where X is the solution of the fractal variational inequality problem G and X^h is the solution of the corresponding discretized problem. If, moreover, $X_j \to X$ strongly in V, then the weak convergence in eqns (34) and (35) is replaced by the strong convergence.

It results from the above propositions that, in order to solve a fractal unilateral problem G, it is sufficient to solve a sequence of classical unilateral problems G_j by the F.E. method or any other discretization procedure. At the limit $j \to \infty$ and $h \to \infty$, the solution of the

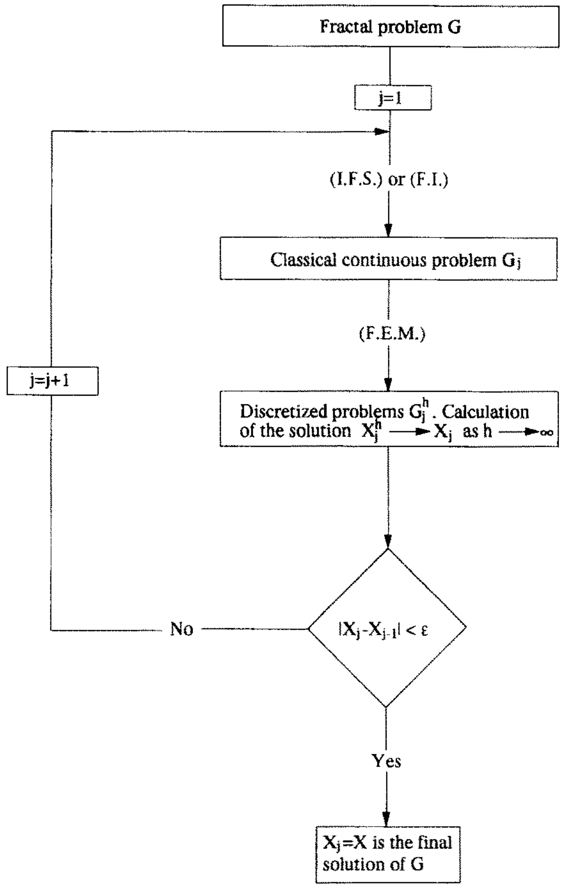


Fig. 1. General flow chart of the fractal approximation algorithm.

fractal variational inequality is obtained. The numerical computer schemes to which all these propositions lead are generally described in Fig. 1.

4. NUMERICAL APPLICATIONS: BILATERAL PROBLEM

As a first application we shall solve the tree-like regular cantilever beam of Fig. 2. Tree-like structures may represent general networks, pipe line systems, natural trees (Prusinkiewicz and Hanan, 1989), human arteries, diffusion-limited aggregation models (Feder, 1988) etc. In our case, the tree-like structure represents a cell, especially a Purkinje cell in the human brain (Muller and Reinhardt, 1990). We consider that this cell is subjected to a uniform temperature change of 3°C. In Fig. 2 we give the displacement field for such a structure. The thermal coefficient of expansion α is taken to be 1.2×10^{-5}. The method of calculation is described in Panagiotopoulos (1992b). The assumption we make here is that the cell branches are beam elements with a modulus of elasticity $E = 10^6$ kN/m^2, Poisson's ratio $\nu = 0.10$ and diameter $d = 4$ mm (the numerical data do not correspond to the reality).

As a second application we consider a spongy body which is simulated by the Sierpinski triangle of Fig. 3. This structure is assumed to be the attractor of a "just touching" I.F.S. (Barnsley, 1988) which is described by the following three contraction mappings:

$$w_1\begin{Bmatrix}x\\y\end{Bmatrix} = \begin{bmatrix}0.50 & 0.0\\0.0 & 0.50\end{bmatrix}\begin{Bmatrix}x\\y\end{Bmatrix} + \begin{Bmatrix}0.0\\0.501_{AC}\end{Bmatrix}$$

$$w_2\begin{Bmatrix}x\\y\end{Bmatrix} = \begin{bmatrix}0.50 & 0.0\\0.0 & 0.50\end{bmatrix}\begin{Bmatrix}x\\y\end{Bmatrix} + \begin{Bmatrix}0.0\\0.0\end{Bmatrix}$$

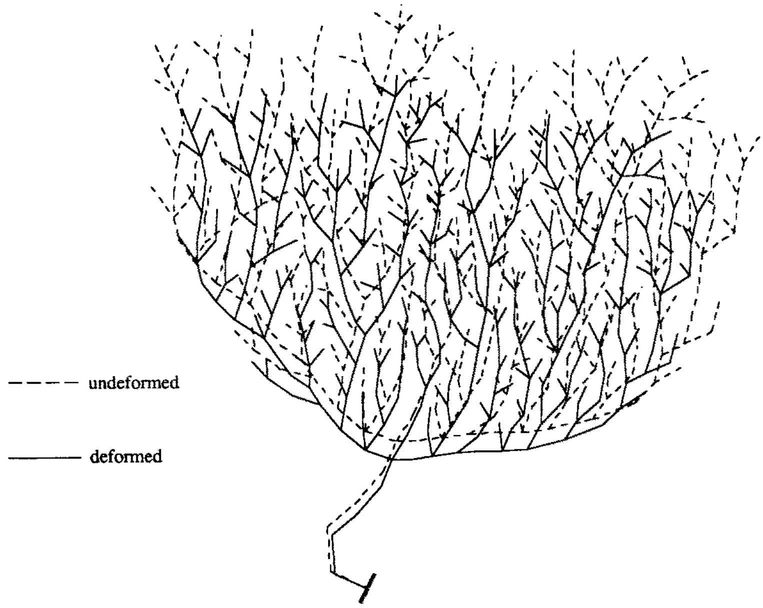


Fig. 2. Displacement field for a tree-like fractal structure.

$$w_3 \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} 0.50 & 0.0 \\ 0.0 & 0.50 \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} + \begin{Bmatrix} 0.50 l_{AB} \\ 0.0 \end{Bmatrix}.$$

The material we have is linear elastic with a modulus of elasticity $E = 2.1 \times 10^6$ kN/m^2, Poisson's ratio $\nu = 0.33$ and thickness 0.15 m. The fractal dimension of the body is $D = \ln 3/\ln 2$. What we study here is the dynamic characteristics-mode shapes (eigenvectors) and frequencies (eigenvalues) for the different approximations $Q_1, Q_2, \dots$ of the fractal body. All the approximations of the fractal structure have been discretized by triangular constant stress (plane strain) elements. It is important to notice that the finite elements used

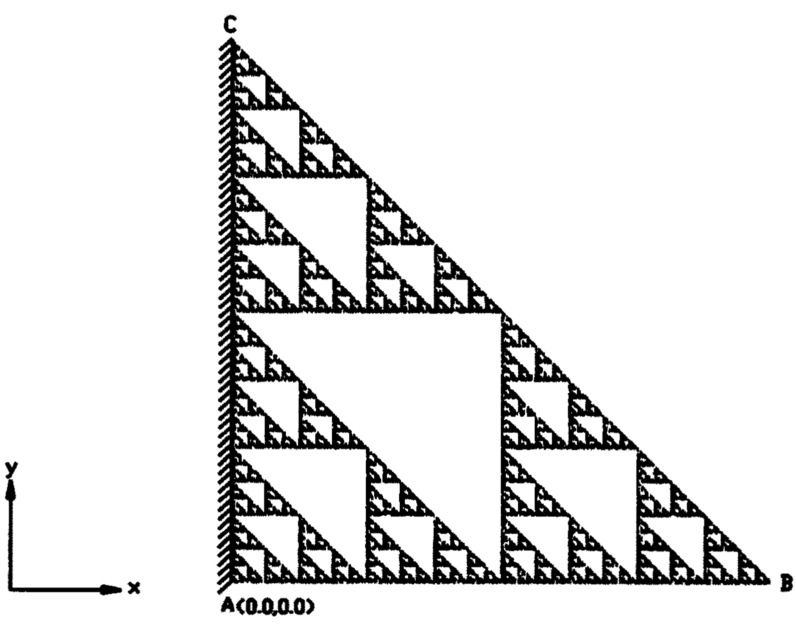


Fig. 3. A spongy body idealized through the Sierpinski triangle.

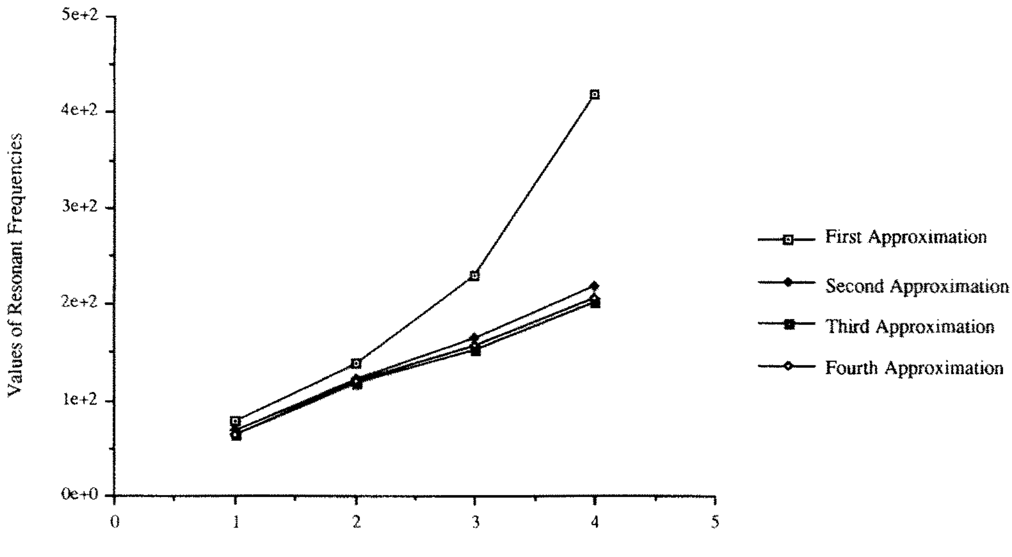


Fig. 4. Values of the first four frequencies for the approximations Q_1 to Q_4.

here are of the same size for all iterations $Q_1, Q_2, \ldots$ of the fractal body. The number of the triangular finite elements which were used for the calculations are 12288, 9216, 6912, 5184 for the first, second, third and fourth approximations, respectively. The dynamic characteristics are computed by the normal mode analysis of the program MARC.

In Fig. 4 the first four frequencies for the approximations Q_1, Q_2, Q_3 and Q_4 of the fractal body are given. From the numerical results (Table 1), the accumulation of the results after the second approximation towards the solution of the problem is evident.

5. NUMERICAL APPLICATIONS: UNILATERAL PROBLEM

In this section we shall study the unilateral contact problem with friction for fractal interfaces. Therefore, we consider a structure with a fractal interface Φ. We assume that Coulomb's friction conditions hold at the interface if debonding does not occur. *A priori* it is not known where debonding occurs. In order to define the interface conditions of the problem, we consider the normal and tangential components S_N, $S_T = \{S_{T_i}\}$ of the boundary tractions $S = \{S_i\}$.

The boundary conditions on the interface are described by the relations:

(i) if $[u_N] < 0$ then $S_N = 0$ and $S_T = 0$, (36)

(ii) if $[u_N] = 0$ then $S_N < 0$,

if $|S_T| < \mu|S_N|$, then $[u_T] = \{[u_{T_i}]\} = 0$, (37)

if $|S_T| = \mu|S_N|$, then there exists $\lambda \geqslant 0$ such that $[u_{T_i}] = -\lambda S_{T_i}$, (38)

$i = 1, 2, 3$.

Here μ is the frictional coefficient; $[u_N]$ is considered to be negative if it tends to open the crack.

Table 1.

Iterations	Resonant frequencies 1	2	3	4
1	78.655	138.633	229.827	419.913
2	67.324	122.133	165.894	218.013
3	64.001	117.885	153.709	203.193
4	65.019	119.624	156.009	206.849

Concerning now the behavior of the body, we assume that it is a linear elastic body obeying Hooke's law and that we have a geometrically linear theory. Let V be the linear space of the displacements v_i, and let V_0 be the set of the kinematically admissible displacements, i.e.

$$V_0 = \{v | v = \{v_i\}, \quad v_i \in V_i, \quad v_i = 0 \quad i = 1,2,3 \text{ on } \Gamma_U\}. \tag{39}$$

The work $\int_\Omega f_i v_i \, d\Omega$ of the force $f = \{f_i\}$ for the displacement $v = \{v_i\}$ is denoted by (f, v) and the bilinear form of elasticity by

$$\alpha(u, v) = (C\varepsilon(u), \varepsilon(v)) = \int_\Omega C_{ijhk}\varepsilon_{ij}(u)\varepsilon_{hk}(v) \, d\Omega. \tag{40}$$

It has been shown by Panagiotopoulos (1975) and Nečas *et al.* (1980) that the position of equilibrium of the structure is obtained in the following way: let us consider first that the tangential forces along the interface are given, i.e.

$$S_{T_i} = C_{T_i}^{(1)}, \quad i = 1,2,3, \text{ on the interface } \Phi. \tag{41}$$

Then we have a simple unilateral contact problem whose solution $u^{(1)}$ is the unique solution of the minimum problem of the potential energy

$$\Pi(u^{(1)}) = \min \{\Pi(v) | v \in V_0, \quad [v_N] \leqslant 0 \text{ on } \Phi\}, \tag{42}$$

where

$$\Pi(v) = \tfrac{1}{2}\alpha(v, v) - (f, v) - \int_{\Gamma_F} F_i v_i \, d\Gamma - \int_\Phi C_{T_i}^{(1)} [v_{T_i}] \, d\Gamma. \tag{43}$$

From $u^{(1)}$ we get the interface normal reactions at the interface S_N, say $C_N^{(1)}$. Let us use the notation $S_N = C_N^{(1)}$ to denote the normal traction to the interface Φ. Then, at the second level we solve the friction problem with given interface normal reactions $S_N = C_N^{(1)}$. In this problem we prefer to minimize the complementary energy of the structure in order to obtain the stress field, because the resulting problem is a classical inequality constrained quadratic programming (Q.P.) problem. We can formulate the same problem as a minimization problem for the potential energy, but in this case a general convex nondifferentiable optimization problem is formulated, since for the friction problem the potential energy contains the friction energy form which is the absolute value function. Thus, in order to avoid the use of the less efficient algorithms of nondifferentiable optimization, we prefer here to minimize the complementary energy. Let us denote by $\tilde{\Sigma}_0$ the statically admissible set

$$\tilde{\Sigma}_0 = \{\tau = \{\tau_{ij}\}, \quad \tau_{ij} = \tau_{ji} \in \Sigma, \quad \tau_{ij,j} + f_i = 0 \quad \text{in } \Omega, \quad T_i = \tau_{ij}n_j = F_i \quad \text{on } \Gamma_F,$$
$$|T_T| \leqslant \mu |C_N^{(1)}| \quad \text{and} \quad T_N = C_N^{(1)} \quad \text{on } \Phi\} \tag{44}$$

where Σ denotes the vector space of the stresses. The unique position of equilibrium $\sigma^{(2)}$ is obtained by the solution of the following minimum problem of the complementary energy.

$$\tilde{\Pi}^c(\sigma^{(2)}) = \min \left\{ \tilde{\Pi}^c(\tau) | \tau \in \tilde{\Sigma}_0 \right\}, \quad \tilde{\Pi}^c(\tau) = \tfrac{1}{2}A(\tau, \tau). \tag{45}$$

From $\sigma^{(2)}$ we obtain the corresponding displacements field $u^{(2)}$ and the interface tangential forces $S_T = C_{T_i}^{(2)}$. This procedure is continued until the differences $|C_{T_i}^{(k+1)} - C_{T_i}^{(k)}|$ and $|C_{N_i}^{(k+1)} - C_{N_i}^{(k)}|$ become appropriately small.

Note at this point that the minimization of the complementary energy requires the formulation, for the chosen finite element scheme, of the force (or compatibility) method. The singular value decomposition is applied for the automatic determination of a solution of the equilibrium equations and for the formulation of the complementary energy (Panagiotopoulos *et al.*, 1992). First the minimum problems for the potential and the complementary energy are formulated for the case of interface with nonfractal geometry. However, in the case of fractured bodies, or metal surfaces subjected to sand-blasting or to meteoritic rain, or interfaces in rock mechanics or biomechanics (bone fracture etc.), fractal interfaces Φ must be taken into account.

Since Φ is the fixed point of a given transformation W, i.e.

$$\Phi = W(\Phi) \quad \text{and} \quad \Phi_{n+1} = W(\Phi_n), \quad \Phi_n \to \Phi \quad \text{when } n \to \infty,$$

we are led, in the case of fractal interfaces, to the following formulation of the unilateral contact and friction problem. Let us consider for the pure unilateral contact subproblem, the potential energy Π_n, in the nth step, over the admissible set V_{0n}. Here Π_n is given by eqn (43), where all the terms are now calculated with respect to the interface Φ_n. Moreover, V_0 does not change for each Φ_n. With respect to Φ_n, the complementary energy $\tilde{\Pi}^c_n$, as well as $\tilde{\Sigma}_{0n}$ for the pure friction subproblem, are formulated.

We denote the corresponding displacement and stress field by u_n and σ_n respectively. Since Φ_n is an interface with classical geometry, the solutions u_n and σ_n are obtained by discretization and solution of the arising minimum problems. We repeat this procedure several times by increasing n and we claim that at the limit $n \to \infty$, u_n and σ_n tend to the solution of the fractal interface problem with contact and friction interface conditions. The convergence of the whole procedure is still an open problem. However the numerical experiments show a quick and stable convergence.

As an example we consider here the structure of Fig. 5 submitted to loading in its plane. We have a linear elastic material with a modulus of elasticity $E = 2.1 \times 10^6$ kN/m^2 and Poisson's ratio $\nu = 0.33$. The thickness is taken as equal to 0.10 m. As is shown in Fig. 5, we have two cracks with fractal geometry, which are defined to be the attractors of the I.F.S.s $\{R^2; w_1, w_2\}$ and $\{R^2; w_3, w_4\}$ where:

$$w_1 \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} 0.70 & 0.00 \\ 0.02 & 0.80 \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} + \begin{Bmatrix} 0.00 \\ 0.00 \end{Bmatrix}$$

$$w_2 \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} 0.30 & 0.00 \\ -0.06 & 0.30 \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} + \begin{Bmatrix} 3.50 \\ 1.70 \end{Bmatrix}$$

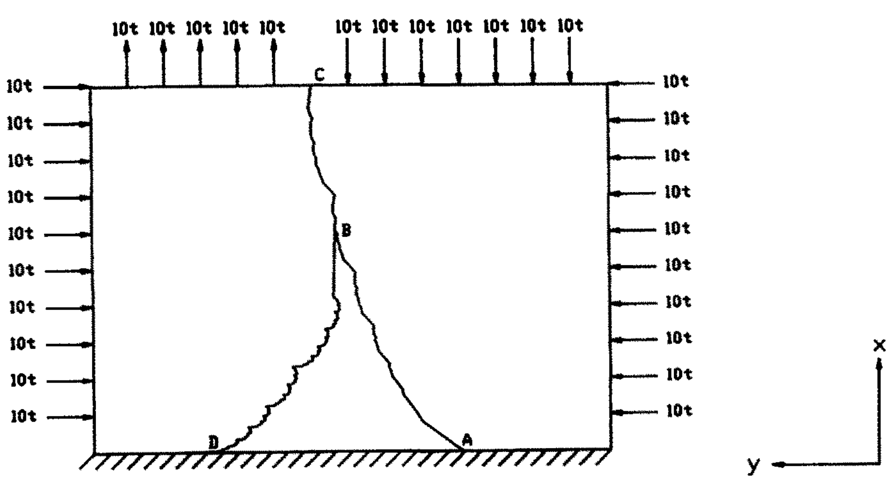


Fig. 5. A multifractured plane elastic structure.

and

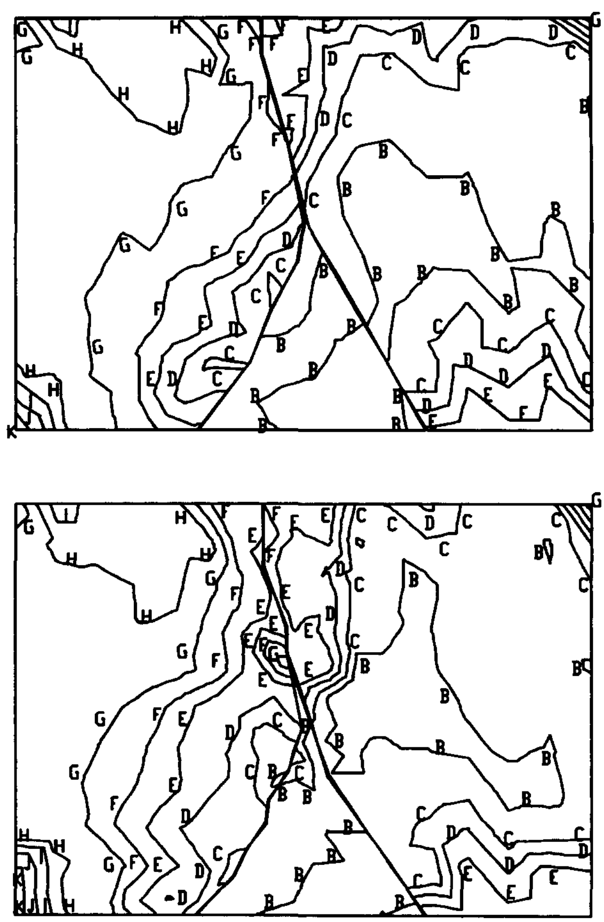
$$w_3\begin{Bmatrix}x\\y\end{Bmatrix}=\begin{bmatrix}0.5714 & 0.00\\0.0543 & 0.80\end{bmatrix}\begin{Bmatrix}x\\y\end{Bmatrix}+\begin{Bmatrix}0.00\\0.70\end{Bmatrix}$$

$$w_4\begin{Bmatrix}x\\y\end{Bmatrix}=\begin{bmatrix}0.4286 & 0.00\\0.1514 & 0.60\end{bmatrix}\begin{Bmatrix}x\\y\end{Bmatrix}+\begin{Bmatrix}2.00\\0.15\end{Bmatrix}.$$

Based on the above I.F.S.s, the different structures corresponding to the consecutive approximations of the fractal interfaces are calculated for the same kinematic conditions and the same loading, and the stress and displacement fields are obtained for each approximation. The previously explained two-level algorithm was applied to analyze the frictional contact problem in each structure on a HP755 workstation. Triangular constant stress elements have been used for the discretization. In Fig. 6 the von-Mises stresses for the approximations f_2 to f_5 are given. We observe that the stress field becomes stable near to

(a)

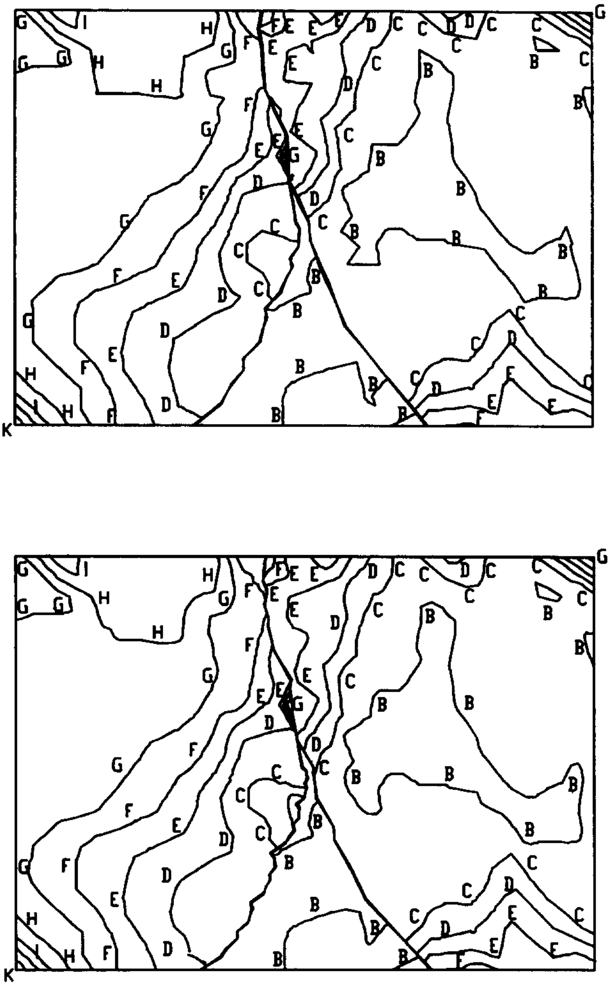
(b)



Von-Mises stress

A 0.0000E+00	E 2.0762E+02	I 4.1524E+02
B 5.1905E+01	F 2.5952E+02	J 4.6714E+02
C 1.0381E+02	G 3.1143E+02	K 5.1905E+02
D 1.5571E+02	H 3.6333E+02	L 5.7095E+02
		M 6.2286E+02

Fig. 6. The von-Mises stress for the (a) second, (b) third, (c) fourth and (d) fifth iterations.



(c)

(d)

Fig. 6 (*continued*).

the interface of the fractal crack defined by the first I.F.S. $\{R^2; w_1, w_2\}$ from the third approximation of the fractal interface. On the contrary, it is obvious that the same stress field becomes stable near to the second crack defined by the I.F.S. $\{R^2; w_3, w_4\}$ after the fourth approximation of the fractal interface. This is due to the fact that this interface has many external and re-entrant corners, which create singularities and make use of higher order approximations necessary in order to have a convergence to the final solution of the problem. Note that the irregularities of a fractal boundary increase when its fractal dimension is not near to its topological dimension. In the example we study here, the fractal dimension of the I.F.S. $\{R^2; w_1, w_2\}$ is $D = 1.10$, whereas the same dimension of the I.F.S. $\{R^2; w_3, w_4\}$ is $D = 1.50$.

Another factor which affects the quick convergence is the loading, which in this example guarantees that the contact and noncontact regions do not change considerably in each approximation. Let us also note here that the fractal boundary does not affect the stress and displacement fields inside the body. Therefore, the St Venant principle for fractal boundaries in problems having unilateral contact and friction effects seems to be valid.

6. NUMERICAL APPLICATIONS: UNILATERAL PROBLEMS WITH LARGE DISPLACEMENTS

In the treatment of the previous section, the theory was formulated under the assumption that the arising displacements and the respective strains were small. Here we will extend the developed approximation scheme for structures with fractal-type interfaces to the more

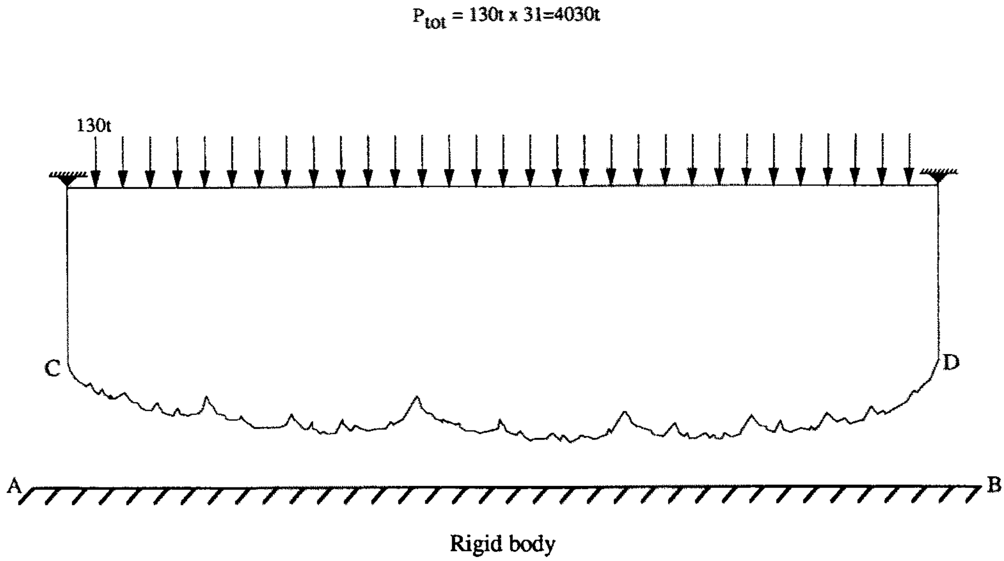


Fig. 7. A structure with a fractal boundary subjected to heavy load in its plane and a rigid body AB.

general case by taking into account our formulation large displacements and large strains. The method applied for the consideration of the geometric nonlinearities is the total Lagrangian method (Bathe, 1982); thus all the quantities are referred to the initial configuration of the body at time zero. Moreover, the second Piola–Kirchhoff stress tensor and the Green–Lagrange strain tensor are used in the F.E. analysis.

In the following, a short description of the classical method for the solution of geometrical nonlinear problems is given. The formulation we use is the total Lagrangian formulation and Newton–Raphson iterations, modified for the solution of unilateral contact problems. Concerning the notation, the left superscripts refer to the considered configuration of the body. Our aim is to express the virtual work in terms of an integral over a known volume and a known surface, and to incrementally decompose the stress and strains in an effective manner. The solution of the problem in the discretized form is obtained through the following iterative scheme:

$$({}^{t}_{0}\mathbf{K}_{\mathrm{L}}+{}^{t}_{0}\mathbf{K}_{\mathrm{NL}})\Delta\mathbf{U}^{i} = {}^{t+\Delta t}\mathbf{R}-{}^{t+\Delta t}_{0}\mathbf{F}^{(i-1)} \tag{46}$$

with

${}^{t}_{0}\mathbf{K}_{\mathrm{L}}$	linear strain incremental stiffness matrix,
${}^{t}_{0}\mathbf{K}_{\mathrm{NL}}$	nonlinear strain (geometric) incremental stiffness matrix,
${}^{t+\Delta t}\mathbf{R}$	vector of externally applied nodal point load at the time $t+\Delta t$,
${}^{t+\Delta t}_{0}\mathbf{F}^{(i-1)}$	vector of nodal point forces equivalent to element stresses at the time $t+\Delta t$ in the $i-1$ iteration (internal forces).
$\Delta\mathbf{U}^{i}$	vector of increments of the nodal point displacements in the i iteration.

$${}^{t+\Delta t}\mathbf{U}^{(i)} = {}^{t+\Delta t}\mathbf{U}^{(i-1)}+\Delta\mathbf{U}^{i}. \tag{47}$$

Here, a left superscript indicates in which configuration the quantity occurs, while a left subscript indicates the configuration with respect to that in which the quantity is measured. In addition, if unilateral contact conditions hold for a part of the boundary, eqn (46) must be written in the form

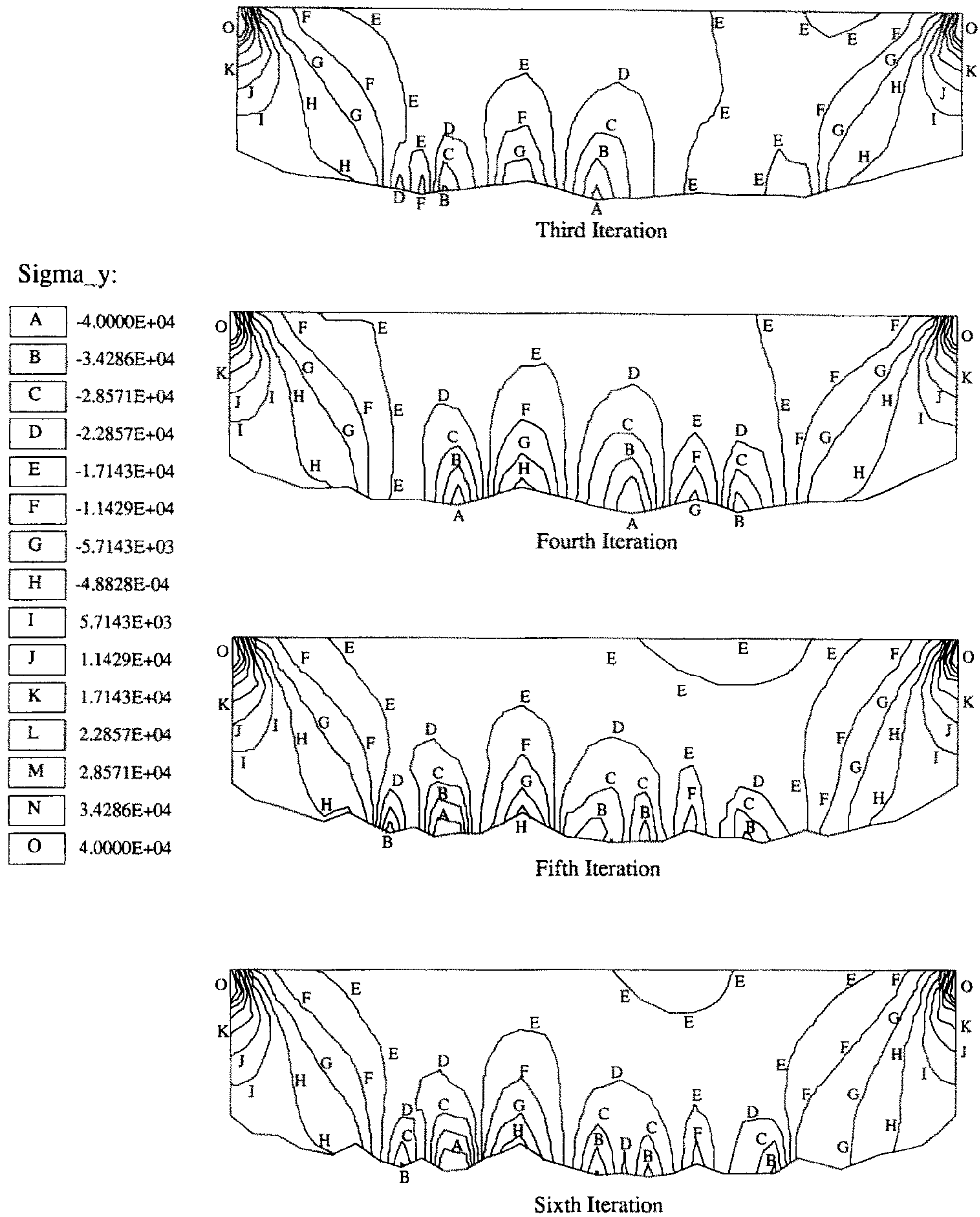


Fig. 8. σ_y stresses from the third to the sixth iteration.

$$({}^t_0\mathbf{K}_{\mathrm{L}} + {}^t_0\mathbf{K}_{\mathrm{NL}})\Delta\mathbf{U}^i = {}^{t+\Delta t}\mathbf{R} - {}^{t+\Delta t}_{0}\mathbf{F}^{(i-1)} \qquad |\mathbf{A}\Delta\mathbf{U}^i > \mathbf{b}^{(i)} \qquad \mathbf{b}^{(i)} = \mathbf{b}^{(i-1)} - \Delta\mathbf{U}^{(i-1)} \tag{48}$$

where $\mathbf{A}$ and $\mathbf{b}^{(i)}$ are appropriately defined matrix and vector introducing the inequality constraints. The solution of every step of the proposed iterative scheme is obtained by applying a simple Q.P. algorithm.

The above analysis was applied to the solution of a 2-D body, which is shown in Fig. 7, with a fractal boundary subjected to heavy load in its plane. Large loads on the upper boundary enforce the fractal part to be in contact with the rigid support AB. The body has been discretized by rectangular isoparametric elements. Linear elasticity and geometrical nonlinearity are assumed. The modulus of elasticity is $E = 2.1 \times 10^6$ kN/m^2 and the Poisson's ratio, $v = 0.33$. The thickness is taken to be 0.05 m. We study the planification of the fractal surface and the corresponding variations of the stress and displacement fields (cf. Fig. 8).

The fractal boundary is defined to be the attractor of the I.F.S. $\{R^2; w_1, w_2\}$ where

$$w_1\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} 0.40 & 0.00 \\ -0.04 & 0.60 \end{bmatrix}\begin{Bmatrix} x \\ y \end{Bmatrix} + \begin{Bmatrix} 0.00 \\ 0.00 \end{Bmatrix}$$

$$w_2\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} 0.60 & 0.00 \\ 0.04 & 0.80 \end{bmatrix}\begin{Bmatrix} x \\ y \end{Bmatrix} + \begin{Bmatrix} 1.00 \\ -0.10 \end{Bmatrix}.$$

We must note here that the above relations describe the I.F.S. on the assumption that the coordinates of the point C are $(x, y) = (0.0, 0.0)$.

With this I.F.S. we calculate the approximations of the fractal interface, and for each approximation the stress and displacement fields are obtained. From the results presented in Fig. 8, we notice that the differences become insignificant after the fifth iteration. This is due to the fact that the Hausdorff distance between the approximation f_5 and the attractor f is very small, i.e. the approximation f_5 sufficiently approximates the attractor f. What we must note here is that in the case of geometrically nonlinear structures, we need higher order approximations than in the case of geometrically linear structures for the same fractal boundary to have convergence.

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APPENDIX: CERTAIN PROOFS OF PROPOSITIONS OF SECTION 3

Proof of Proposition 3.1

From eqns (18) to (22) we obtain

$$a\|X_j\|^2 \leqslant \dot{W}(G_j, X_j) \leqslant \dot{W}(G_j, X^*) + \Phi(G_j, X^*) - \Phi(G_j, X_j) - (\dot{p}(G), X^* - X_j) \quad \forall X^* \in V. \tag{A1}$$

This relation is now combined with eqn (23). Due to eqns (26) and (27), we can assume that $\dot{W}(G_j, X^*) < \infty$ and $\Phi(G_j, X^*) < \infty$. We have that (c denotes the various constants)

$$c\|X_j\|^2 \leqslant c\|X_j\| + c + \|\dot{p}(G_j)\| \, \|X^* - X_j\|. \tag{A2}$$

Then eqn (A2) implies with eqn (3) that $\|X_j\| < c_1$ where c_1 is a constant. Thus, a subsequence of X_j can be determined such that

$$X_j \to X \text{ weakly in } V. \tag{A3}$$

Now let us write eqn (20) in the form

$$\dot{W}(G_j, X_j) + \Phi(G_j, X_j) \leqslant \dot{W}(G_j, X^*) + \Phi(G_j, X^*) - (\dot{p}(G_j), X^* - X_j) \quad \forall X^* \in V. \tag{A4}$$

Taking the liminf from both sides in (A4) we obtain, using eqns (3) and (24)–(27) and observing that the liminf of the r.h.s. is equal to the limit because this limit exists, that

$$\Phi(G, X) + \dot{W}(G, X) \leqslant \liminf\,(\Phi(G_j, X_j) + \dot{W}(G_j, X_j)) \leqslant \dot{W}(G, X^*) + \Phi(G, X^*) - (\dot{p}(G), X^* - X) \quad \forall X^* \in V. \tag{A5}$$

Thus X is a solution of eqn (28).

Proof of Proposition 3.3

In this case eqn (20) takes the form (Panagiotopoulos, 1985): find $X_j \in V$ so as to satisfy the variational inequality

$$(L(G_j)X_j, X^* - X_j) + \Phi(G_j, X^*) - \Phi(G_j, X_j) \geqslant (p(G_j), X^* - X_j) \quad \forall X^* \in V. \tag{A6}$$

Here

$$(L(G_j)X_j, X^*) = \int_{\Omega_j} C_{iklm}\varepsilon_{ik}(X_j)\varepsilon_{lm}(X^*)\, d\Omega. \tag{A7}$$

Let us now put $X^* = X$ into eqn (A6). Then from eqns (32) and (A7) we obtain, using eqn (A6), the relation

$$\begin{aligned} c\|X - X_j\|^2 &\leqslant (L(G_j)(X - X_j), X - X_j) = (L(G_j)X, X - X_j) - (L(G_j)X_j, X - X_j) \\ &\leqslant (L(G_j)X, X - X_j) + \Phi(G_j, X) - \Phi(G_j, X_j) - (p(G_j), X - X_j). \end{aligned} \tag{A8}$$

But the relation in eqn (24) implies that for $X_j \to X$ weakly in V

$$\limsup\, -\Phi(G_j, X_j) \leqslant -\Phi(G, X). \tag{A9}$$

Thus, from eqns (A9), (27) and (3), we have by taking the limsup of both sides of eqn (A8) that

$$\limsup\, \|X - X_j\|^2 \leqslant 0. \tag{A10}$$

Here we have also used the fact that $L(G_j) = \operatorname{grad} \dot{W}(G_j)$. This relationship, together with eqn (26) implies, due to the definition of gradient, that $L(G_j)X \to L(G)X$ strongly in V for every $X \in V$.

Accordingly we have that $\|X - X_j\| \to 0$.

Proof of Proposition 3.4

Due to the assumptions of proposition 3.1 we have the convergence

$$X_j^h \to X^h \quad \text{in } V^h \tag{A11}$$

(weak and strong convergence are identical since V^h is finite dimensional) and

$$X_j \to X \text{ weakly in } V \text{ as } j \to \infty. \tag{A12}$$

Thus for any linear functional l we may write that

$$|l(X - X_j^h)| = |l(X - X_j) + l(X_j - X_j^h)| \leqslant |l(X - X_j)| + |l(X_j - X_j^h)|. \tag{A13}$$

The r.h.s. tends to zero because of eqns (A12) and (33). Similarly

$$|l(X - X^h)| \leqslant |l(X - X_j^h)| + |l(X_j^h - X^h)| \to 0. \tag{A14}$$

Thus the weak convergence is proved. In order to show the strong convergence we proceed as in eqns (14) and (15).